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On the numerical radius of some C_{00}-contractions

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We prove the Haagerup–de la Harpe type inequalities for numerical radius of some Hilbert space C_{00}-contractions.

1. Introduction

Let H be a complex Hilbert space, with inner product $\langle , \rangle$, and denote by $\mathcal{B}(H)$ the algebra of bounded linear operators on H. Given an operator $T \in \mathcal{B}(H)$, we denote as usual by

$$W(T) = \{\langle Tx, x\rangle : x \in (H)_1\},$$

the numerical range of T, where $(H)_1 = \{x \in H : \|x\| = 1\}$ is the unit sphere in H, and by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\},$$

the numerical radius of T. Clearly, $w(T) \leqslant \|T\|$. Also, it follows from polarization (see [1]) that

$$w(T) \geqslant \frac{1}{2}\|T\| \tag{1}$$

for any $T \in \mathcal{B}(H)$.

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Haagerup and de la Harpe [2] proved the following inequality for the numerical radius of a nilpotent contraction $N \in \mathcal{B}(H)$:

$$w\,(N) \leqslant \cos\frac{\pi}{n+1}, \tag{2}$$

where n, is the nilpotency degree of N, that is $N^n = 0$, but $N^{n-1} \neq 0$. In what follows another proofs of Haagerup and de la Harpe's inequality (2) are given by many authors. We refer to [3–7] for recent papers related to the inequality (2).

In the present paper we prove the Haagerup–de la Harpe type inequality for the numerical radius of some C_{00}-contractions, beyond the class of nilpotents.

Recall that C_{00} is the class of all contractions $T \in \mathcal{B}(H)$, for which $s - \lim_n T^n = s - \lim_n T^{*^n} = 0$.

It is well-known result of Sz.-Nagy and Foias (see [8,9]) that each $T \in C_{00}$ is unitary equivalent to its model operator $M_\theta = P_\theta S_E | K_\theta$, acting on the model space $K_\theta = H^2(E) \ominus \theta H^2(E)$. Here $\theta = \theta_T \in H^\infty(\mathcal{B}(E))$ is the characteristic function of the contraction T (a two-sided inner function); E is a Hilbert space with $\dim E = \dim(I - T^*T)H, H^2(E)$ is the Hardy space of E-valued functions consisting of all Taylor series $f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n, z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\hat{f}(n) \in E,\ n \geqslant 0, \sum_{n=0}^\infty \|\hat{f}(n)\|_E^2 < +\infty;\ P_\theta = I - \theta P_+ \theta^*$ is the orthogonal projection of $H^2(E)$ onto K_θ, where P_+ is the Riesz projector of $L^2(E)$ onto $H^2(E)$, and $S_E f = zf$ is the unilateral shift operator on $H^2(E)$.

We refer to [8–10] for more detail information on the model theory of Sz. Nagy and Foias, theory of Hardy spaces and Hankel operators.

2. Results

1. In the following theorem we use the Sz.-Nagy–Foias model operators and Hankel operator techniques to obtain the Haagerup–de la Harpe type inequality for some C_{00} contractions. Before formulating the theorem, note that if x is a vector in $H^2(E)$, then in its inner-outer factorization $x = x_i x_e$ the outer factor x_e is a scalar function, $|x_e(z)| = \|x(z)\|_E$, and correspondingly, $x_i(z) = \frac{x(z)}{x_e(z)}$ is a vector from $H^2(E)$ possessing unit norm in E almost everywhere on the unit circle $\mathbb{T}$.

Theorem 1. *Let Θ be a two-sided inner function, $K_\Theta = H^2(E) \ominus \Theta H^2(E)$ (E is some auxiliary Hilbert space) be a model space and $M_\Theta = P_\Theta S_E | K_\Theta$ be a corresponding model operator. Suppose that there exist $x \in (K_\Theta)_1 \cap H^\infty(E)$ and an integer $n \geqslant 1$ such that:*

(a) $S^{*n} x_e \in H^\infty(\mathbb{D})$, where S is the shift operator on the Hardy space $H^2 = H^2(\mathbb{D}), Sf = zf$;
(b) $\langle M_\Theta^k x, x\rangle = 0$ for each $k \geqslant n$;
(c) $|\langle M_\Theta x, x\rangle| = w(M_\Theta)$.
Then $w(M_\Theta) \leqslant \cos\frac{\pi}{n+1}$.

Proof. Let $x = x_i x_e$ be the inner–outer factorization of the function $x \in (K_\Theta)_1 \cap H^\infty(E)$. Since x_i is an inner function, condition (b) implies that

$$\left\langle z^k \sum_{i \geqslant o} \hat{x}_e\,(i)\, z^i, \sum_{i \geqslant o} \hat{x}_e\,(i)\, z^i \right\rangle = 0 \quad (k \geqslant n)$$

That is

$$\sum_{i \geqslant o} \hat{x}_e\,(i)\, \overline{\hat{x}_e\,(i+k)} = 0 \quad (k \geqslant n)$$

or

$$\begin{aligned} &\textstyle\sum_{i \geqslant o} \hat{x}_e\,(i)\, \overline{\hat{x}_e\,(i+n)} = 0, \\ &\textstyle\sum_{i \geqslant o} \hat{x}_e\,(i)\, \overline{\hat{x}_e\,(i+n+1)} = 0, \\ &\dots \end{aligned}$$

i.e.,

$$\begin{pmatrix} \overline{\hat{x}_e\,(n)} & \overline{\hat{x}_e\,(n+1)} & \cdots \\ \hat{x}_e\,(n+1) & \hat{x}_e\,(n+2) & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \hat{x}_e\,(0) \\ \hat{x}_e\,(1) \\ \vdots \end{pmatrix} = 0.$$

Denoting

$$\mathcal{H}_n \stackrel{def}{=} \begin{pmatrix} \overline{\hat{x}_e\,(n)} & \overline{\hat{x}_e\,(n+1)} & \cdots \\ \hat{x}_e\,(n+1) & \hat{x}_e\,(n+2) & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}, \quad \overrightarrow{x} \stackrel{def}{=} \begin{pmatrix} \hat{x}_e\,(0) \\ \hat{x}_e\,(1) \\ \vdots \end{pmatrix}$$

we have

$$\mathcal{H}_n \overrightarrow{x} = 0, \tag{3}$$

where $\mathcal{H}_n$ is an infinite Hankel matrix. We have

$$S^{*n} x_e\,(z) = \sum_{k \geqslant o} \hat{x}_e\,(n+k)\, z^k \stackrel{def}{=} \varphi_n\,(z) \in H^\infty\,(\mathbb{D}) \quad \text{(by condition } (a)\text{)}.$$

Obviously,

$$\overline{\hat{x}_e\,(n+(k+m))} = \hat{\varphi_n^*}\,(k+m) \quad (k \geqslant 0, m \geqslant 0)\,,$$

where $\varphi_n^*(z)$ denotes the function $\overline{\varphi_n(\overline{z})}$. Then by Nehari's theorem [11] it is clear from the condition (a) of the theorem that the Hankel operator $\Gamma_{\varphi_n^*}$,

$$\Gamma_{\varphi_n^*} f = \sum_{k,m \geqslant o} \hat{\varphi_n^*}\,(k+m)\, \hat{f}\,(k)\, z^k, \quad f \in H^2$$

is a continuous operator in the space H^2, i.e., the Hankel matrix $\mathcal{H}_n$ determines the continuous Hankel operator $\Gamma_{\varphi_n^*}$ in the space H^2. Therefore the equality (3) means that

$$\Gamma_{\varphi_n^*} x_e = 0, \tag{4}$$

that is, $x_e \in \ker \Gamma_{\varphi_n^*}$. Since the Hankel operator $\Gamma_{\varphi_n^*}$ satisfies the equality

$$\Gamma_{\varphi_n^*} S = S^* \Gamma_{\varphi_n^*},$$

$\ker \Gamma_{\varphi_n^*} \in LatS$, i.e., $\ker \Gamma_{\varphi_n^*}$ is S-invariant subspace. Then according to the theorem of Beurling [10] there exists an inner function $\Omega \in H^\infty(\mathbb{D})$ such that $\ker \Gamma_{\varphi_n^*} = \Omega H^2$. Consequently, equality (4) means that $x_e \in \Omega H^2$. But, since $x_e \in H^2$ is an outer function, the latter inclusion is possible only in the case when $\Gamma_{\varphi_n^*} = 0$. Therefore, $0 = \hat{x_e}(n) = \hat{x_e}(n+1) = \cdots$, and thus, x_e is a polynomial with degree $\leqslant n-1$:

$$x_e\,(z) = \hat{x_e}\,(0) + \hat{x_e}\,(1)\, z + \cdots + \hat{x_e}\,(n-1)\, z^{n-1}.$$

Then

$$\begin{aligned} \langle M_\Theta x, x\rangle &= \langle P_\Theta z x_i x_e, x_i x_e\rangle \\ &= \langle z x_i x_e, x_i x_e\rangle = \langle z x_e, x_e\rangle \\ &= \sum_{i=0}^{n-2} \hat{x_e}\,(i)\, \overline{\hat{x_e}\,(i+1)}. \end{aligned}$$

From this we deduce that

$$|\langle M_\Theta x, x\rangle| \leqslant \sum_{i=0}^{n-2} \left|\hat{x_e}\,(i)\right| \left|\hat{x_e}\,(i+1)\right| \leqslant \cos \frac{\pi}{n+1}.$$

Consequently, by virtue of condition (c) of the theorem we have

$$w(M_\Theta) = |\langle M_\Theta x, x\rangle| \leqslant \cos \frac{\pi}{n+1},$$

which completes the proof. $\square$

Let us give a nontrivial example satisfying conditions of Theorem 1.

Example 1. Let S, $Sf(z) = zf(z)$, be a shift operator acting in the Hardy space $H^2(\mathbb{D})$, $N_S := S(I - SS^*)$, and let T be a contraction on the Hilbert space H such that $\|T\| < \frac{1}{2}$. Let us consider the following operator on the space $H^2 \oplus H$:

$$A := N_S \oplus T.$$

Since $\|N_S\| = 1$, $N_S^2 = 0$, $T \in C_{00}$, we have that $A \in C_{00}$ with $\|A\| = 1$. On the other hand, it is easy to verify that $|\langle N_S f, f\rangle| = |\hat{f}(0)\overline{\hat{f}(1)}| \leqslant \frac{1}{2}$ for every $f \in (H^2)_1$. Since $w(T) < \frac{1}{2}$, clearly the numerical radius $w(A)$ of the operator A is attained in the element $\frac{1+z}{\sqrt{2}} \oplus 0$. Obviously, $\frac{1+z}{\sqrt{2}} \oplus 0 \in \ker(A^2)$, and consequently $\left\langle A^k \frac{1+z}{\sqrt{2}} \oplus 0, \frac{1+z}{\sqrt{2}} \oplus 0\right\rangle = 0$ for all $k \geqslant 2$. Moreover, $\frac{1+z}{\sqrt{2}}$ is an outer function and $S^{*^2} \frac{1+z}{\sqrt{2}} = 0$, and thus A satisfies the conditions of Theorem 1, which gives that $w(A) \leqslant \frac{1}{2}$.

2. Our next result is a slight generalization of Haagerup–de la Harpe 's result in [2] which uses the classical Fejer theorem (see [12]) on positive trigonometric polynomials. Its proof is similar to the proof in [2].

Theorem 2. *Let $T \in \mathcal{B}(H)$ be a contraction such that $\sigma(T) \subset \mathbb{D}$. Suppose that there exist $x \in (H)_1$ and an integer $n \geqslant 1$ such that*:

(*a*) $\langle T^k x, x\rangle = 0$ *for each* $k \geqslant n$;
(*b*) $|\langle Tx, x\rangle| = w(T)$
Then $w(T) \leqslant \cos \frac{\pi}{n+1}$.

Proof. Let us set

$$f_x(t) = \sum_{k=-(n-1)}^{n-1} f_{x,k} e^{ikt},$$

where

$$f_{x,k} = \begin{cases} 1, & \text{if } k = 0, \\ \langle T^k x, x\rangle, & \text{if } k > 0, \\ \langle x, T^{|k|} x\rangle, & \text{if } k < 0. \end{cases}$$

Since $\sigma(T) \subset \mathbb{D}$, for each $t \in \mathbb{R}$ $I - e^{it}T$ is an invertible operator. Then by condition (a) we have

$$\begin{aligned} f_x(t) &= \left\langle \left[I + \sum_{k \geqslant 1} \left(e^{it}T\right)^k + \left(e^{-it}T^*\right)^k \right] x, x \right\rangle \\ &= \left\langle \left[\left(I - e^{it}T\right)^{-1} + \left(I - e^{-it}T^*\right)^{-1} - I \right] x, x \right\rangle \\ &= \left\langle \left(I - e^{it}T^*\right)^{-1} \left[I - e^{-it}T^* + I - e^{it}T - \left(I - e^{-it}T^*\right)\left(I - e^{it}T\right) \right] \right. \\ &\quad \left. \cdot \left(I - e^{it}T\right)^{-1} x, x \right\rangle \end{aligned}$$

$$= \left\langle \left(I - e^{it}T^*\right)^{-1} (I - T^*T) \left(I - e^{it}T\right)^{-1} x, x \right\rangle$$
$$= \left\langle (I - T^*T) \left(I - e^{it}T\right)^{-1} x, \left(I - e^{it}T\right)^{-1} x \right\rangle.$$

Since $\|T\| \leqslant 1$, the inequality $I - T^*T \geqslant 0$ is valid, and hence, the last equality means that $f_x(t) \geqslant 0$, i.e., the trigonometric polynomial $f_x(t)$ is positive. Then by the Fejer theorem, we have

$$|f_{x,1}| \leqslant f_{x,0} \cos \frac{\pi}{n+1},$$

that is,

$$|\langle Tx, x\rangle| \leqslant \cos \frac{\pi}{n+1}.$$

Thus, by condition (b) we have

$$w(T) \leqslant \cos \frac{\pi}{n+1},$$

as desired. □

The following is an immediate corollary of Theorem 2, which gives a bound better than the Haagerup-de la Harpe theorem.

Corollary 3. *Suppose* $T \in \mathcal{B}(H)$ *satisfies* $\|T\| \leqslant 1, T^3 = 0$ *and there exists* $x \in (H)_1$ *such that* $\langle T^2x, x\rangle = 0$ *and* $|\langle Tx, x\rangle| = w(T)$. *Then* $w(T) \leqslant \frac{1}{2}$.

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References

[1] P.R. Halmos, A Hilbert Space Problem Book, Graduate Texts in Mathematics, second ed., Springer-Verlag, New York, 1982.
[2] U. Haagerup, P. de la Harpe, The numerical radius of a nilpotent operator on a Hilbert space, Proc. Amer. Math. Soc. 115 (1992) 371–379.
[3] P.Y. Wu, Unitary dilations and numerical ranges, J. Oper. Theory 38 (1997) 25–42.
[4] C. Pop, On a result of Haagerup–de la Harpe, Revue Roum. Math. Pures Appl. 93 (1998) 9–10.
[5] C.Y. Suen, W_A contractions, Positivity 2 (1998) 301–310.
[6] C. Badea, G. Cassier, Constrained von Neumann inequalities, Adv. Math. 166 (2002) 260–297.
[7] M.T. Karaev, New proofs of the Haagerup–de la Harpe inequality, Math. Notes 75 (2004) 731–733; translated from Matematicheskie Zametki 75 (2004) 787–788.
[8] B. Sz.-Nagy, C. Foias, Harmonic Analysis of Operators on Hilbert Space, North-Holland, Amsterdam, 1970.
[9] N.K. Nikolski, Treatise on the Shift Operator, Springer-Verlag, Heidelberg, 1986.
[10] K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood, Cliffs, NJ, 1962.
[11] Z. Nehari, On bounded bilinear forms, Ann. Math. 65 (1957) 153–162.
[12] G. Polya, G. Szegö, Problems and Theorems in Analysis, vol. II, Springer-Verlag, Berlin, 1976.